University of California, Davis
Society of Manufacturing Engineers at UC Davis

EME 172 and EEC 157A* Control Systems Unofficial

Unofficial Quiz - Solutions

*Disclaimer: This document is a quiz that focuses on control systems theory. It is an unofficial quiz and does not necessarily reflect the format—in the length of any exams, content covered, the protocol, and other aspects—of an actual final exam of EEC 157A or EME 172 in University of California, Davis. However, this covers multiple topics that seems to be a complete agenda of those courses and this document is an attempt to give students extra practice. The problems in this document are written entirely by the author. Any similarity, either in part or in whole, is a complete coincidence. If an error is caught, or if you have any questions and inquiries, please contact the author at mnhyu@ucdavis.edu.

This quiz has eight (8) pages, including this front cover page and the formula sheet.

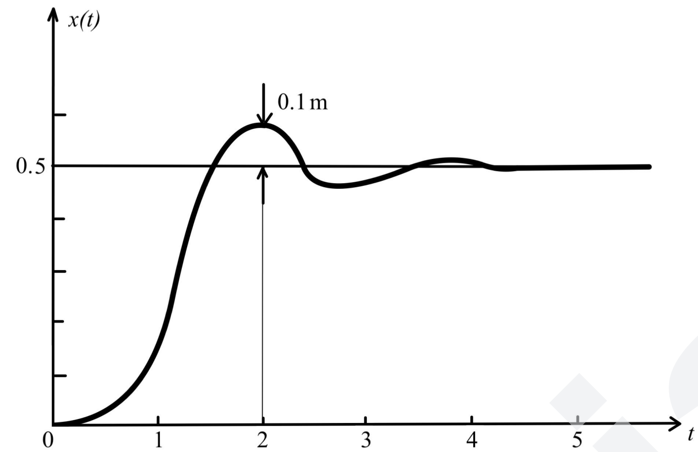
Although there are significant overlaps, some topics may not necessarily be in EEC 157A or EME 172, whichever you may be taking. Do not feel pressured to study those not covered.

Question 1) Time Domain Analysis	
Question 2) Root Loci	
Question 3) Nyquist Criterion	
Question 4) Controllers	

Note: j is used to represent the square root of -1, and not i.

1) Time Domain Analysis

An open loop response, after applying a step function, is shown below (not drawn to scale)



a) Find a numerical expression for $G(s)$, which is a second order transfer function that relates the output response over the input.
b) Find $x(t)$ and the 96% settling time.

$\%OS = 20\%$ $\quad \frac{0.5+01}{0.5} - 1 = 0.2 \longrightarrow \zeta = \frac{-\ln 0.2}{\sqrt{\pi^2 + (\ln 0.2)^2}} \simeq 45.59\%$

$T_p = 2s = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \longrightarrow \omega_n = \frac{\pi}{2\sqrt{1-(.4559)^2}} = 1.7649 \text{ rad/sec}$

so far, we have $G(s) = K\frac{1.7649^2}{s^2 + 2(1.765 * .4559)s + 1.7649^2} \simeq K\frac{3.115}{s^2+1.609s+3.115}$

$\lim_{s\to 0} \frac{s}{s} G(s) = 0.5$ (final theorem for a step response) Intuitively, $K = 0.5$ to satisfy.

$\therefore \boxed{G(s) \simeq \frac{1.557}{s^2+1.609s+3.115}}$ $\quad$ poles: $\frac{-1.609}{2} \pm \frac{\sqrt{1.609^2 - 4(3.115)}}{2} = -0.8047 \pm j0.7854$

$\frac{G(s)}{s} = X(s) \equiv \frac{A}{s+0.8047+j0.7854} + \frac{B}{s+0.8047-j0.7854} + \frac{C}{s}$

$A = \frac{1.557}{s(s+0.8047+j0.7854)}\Big|_{s=-.8047+j.7854} = \frac{1.554}{(-.8047+j.7854)(2j.7854)} = -0.6157 + 0.6308j$

$B = \frac{1.557}{s(s+0.8047-j0.7854)}\Big|_{s=-.8047-j.7854} = \frac{1.554}{(-.8047-j.7854)(-2j.7854)} = -0.6157 - 0.6308j$

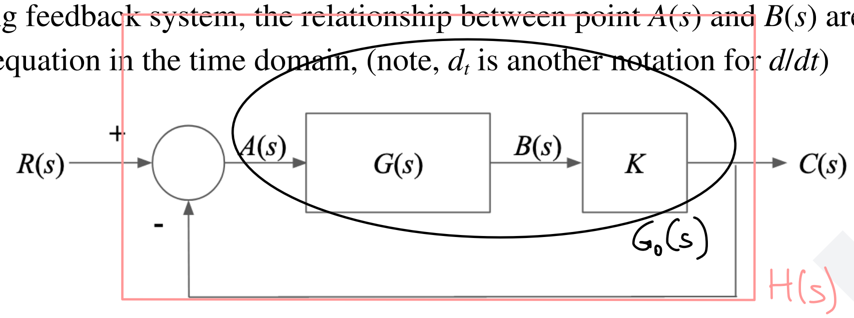
$C = \frac{1.557}{s^2+1.609s+3.115}\Big|_{s=0} = 1/2$

$X(s) \Rightarrow x(t) = \left[(-0.6157+0.6308j)e^{-.8047t}e^{-j.7854t} + (-0.6157-0.6308j)e^{-.8047t}e^{+j.7854t} + \frac{1}{2}\right]u(t)$

$T_s = -\frac{\ln(0.04\sqrt{1-\zeta^2})}{\zeta\omega_n} = \frac{-\ln(0.04\sqrt{1-0.4559^2})}{.4559*1.765} \simeq 4.1448s \simeq T_s$

2) Root Loci

In the following feedback system, the relationship between point $A(s)$ and $B(s)$ are described by the following equation in the time domain, (note, d_t is another notation for d/dt)

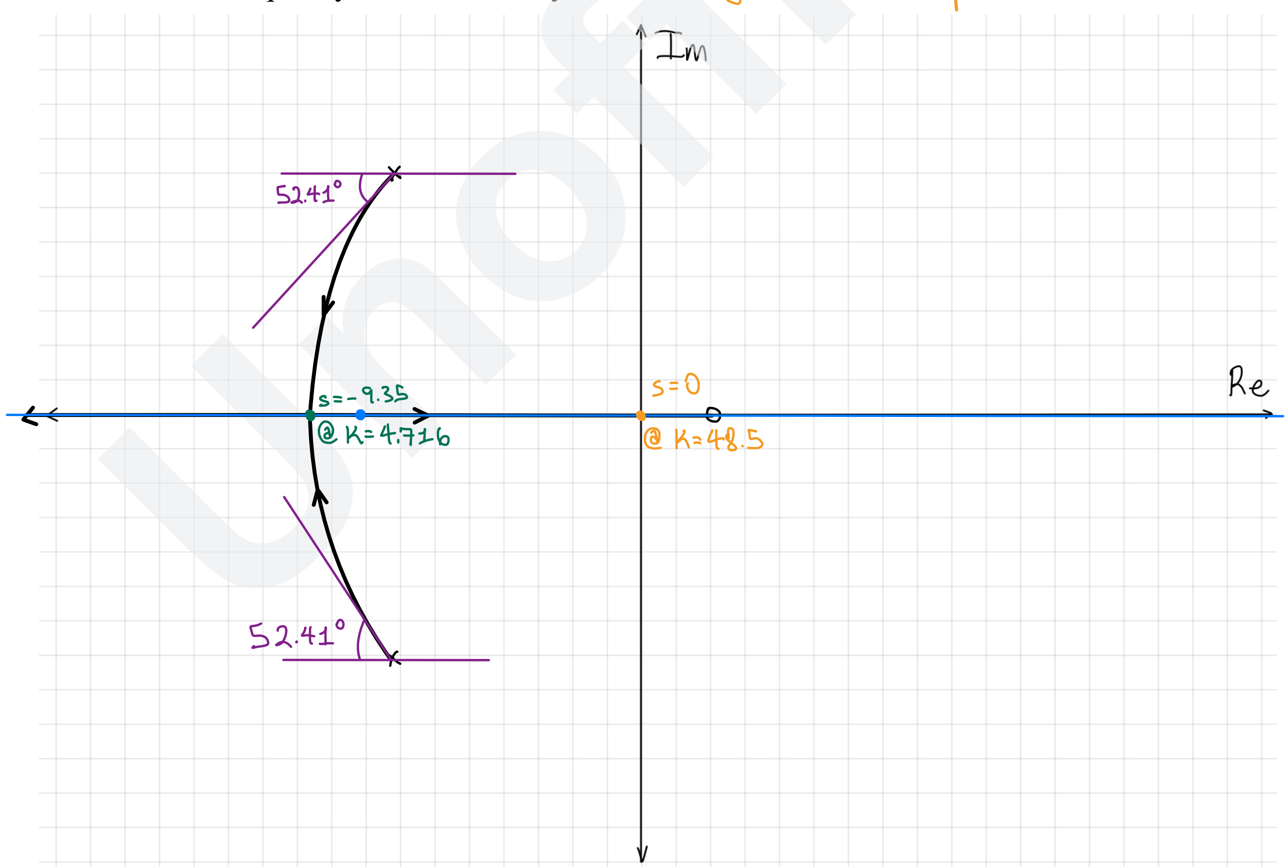


$$d_t^2\, b(t) + 14\, d_t b(t) + 97\, b(t) = d_t a(t) - 2\, a(t)$$

a) Completely sketch the root locus for this system, $0 \le K \le \infty$. Numerically mark
   i) The critically damped point(s) and its/their corresponding K's, if any.
   ii) The closed loop pole(s) where the plot crosses jω axes and its/their corresponding K's.
   iii) The angle of departure from complex poles, if any.
   iv) The asymptote(s) as s approach extremely high frequencies.

b) For part a) i), find $c(t = \infty)$ for a step response. -0.1107

c) For part a) ii), if we apply a step response as an input, does the system oscillate? If so, find its oscillation frequency. Otherwise, why not? No. System is over damped.



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Extra space for question 2)

$$d_t^2 b(t) + 14 d_t b(t) + 97 b(t) = d_t a(t) - 2a(t)$$

$\Downarrow$ Laplace

$$s^2 B(s) + 14 s B(s) + 97 B(s) = sA(s) - 2A(s)$$

$$\frac{B(s)}{A(s)} = \frac{s-2}{s^2+14s+97} = G(s)$$

Zero: $s = 2$ (and ∞)

poles: $\frac{-14}{2} \pm \frac{\sqrt{14^2 - 4 \times 97}}{2}$ or $-7 \pm \underbrace{4\sqrt{3}}_{\sim 6.93} j$

Break-in/away points

$$G_o(s) = K \frac{s-2}{s^2+14s+97}$$

$$d_s G_o(s) = K(s^2+14s+97)^{-1} - K(s-2)(s^2+14s+97)^{-2}(2s+14) = 0$$

$$\cancel{s^2} + \cancel{14s} + 97 = \cancel{2}s^2 - 4s + \cancel{14s} - 28$$

$$0 = s^2 - 4s - 125$$

$$s = \frac{+4 \pm \sqrt{16+500}}{2}$$

$2 \pm \sqrt{4+125} = -9.358$ & $\cancel{13.358}$

Corresponding K: closed loop H(s) denominator

$$\boxed{s^2 + s(14+K) + 97 - 2K} = (s - 2 + \sqrt{129})^2$$

Thus: $(-2+\sqrt{129})^2 = 97 - 2K$

$$(-0.5)\left((-2+\sqrt{129})^2 - 97\right) = K = 4.716$$

Asymptote

$$\sigma_A = \frac{-7+j4\sqrt{3} - 7 - j4\sqrt{3} - 2}{2-1} = -8$$

$$\frac{(\text{sum of finite poles} - \text{sum of finite zeros})}{\#\text{ of finite poles} - \#\text{ of finite zeros}}$$

$$\theta_A = \frac{180 \times (2n+1)}{2-1} = 180^\circ$$

Angle of departure

Let θ_1 be θ from $s_z = 2$ to $s_p = -7 + j4\sqrt{3}$

$\theta_1 = \arctan\left(\frac{4\sqrt{3}}{-9}\right) \simeq -37.6$

θ_2 be θ from the two poles: $90°$

θ_3 is θ of departure

$\theta_3 = 180° + \theta_1 - \theta_2 = 180 + \arctan\left(\frac{4\sqrt{3}}{-9}\right) - 90 \simeq 52.41°$

$$\lim_{s \to 0} \cancel{s} \frac{1}{\cancel{s}} \frac{4.716(s-2)}{s^2 + s(14+K) + 97 - 2(4.716)}$$

(step response)

$$\frac{4.716(-2)}{97 - 2(4.716)} = -0.1077 = c_{step}(t \to \infty)$$

Marginal stability ($j\omega$-axis crossing)

Since closed loop H(s) denominator is

$$s^2 + s(14+K) + \underbrace{97 - 2K}_{\text{this should be } 0}$$

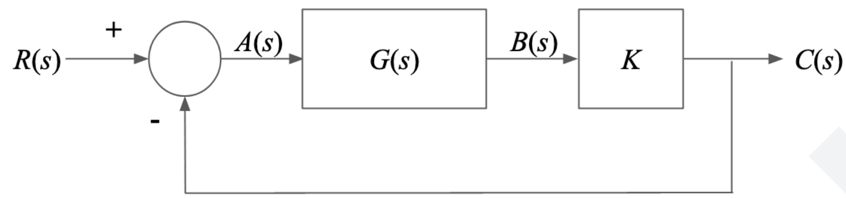
$97 - 2K = 0$, so $K = 97/2 = 48.5$

Does not oscillate since response is overdamped when $4.716 < K < 48.5$.

When $K > 48.5$, $\lim_{t \to \infty} c(t) = -\infty$ (unstable!)

3) Nyquist Criterion

In the following feedback system, the relationship between point $A(s)$ and $B(s)$ are described by the following equation in the time domain, (note, d_t is another notation for d/dt)



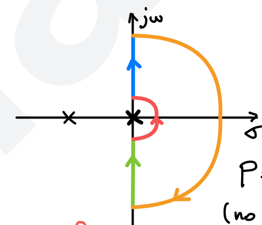
$$d_t^4\, b(t) + 10\, d_t^3\, b(t) = a(t)$$

Completely sketch the Nyquist plot. What values of K, if any, would this system be stable?

Laplace

$$B(s^4 + 10s^3) = A$$

$$\frac{B}{A}(s) = \frac{1}{s^3(s+10)}$$



P=0 (no poles in the plane)

$s = j\omega$

$$G(j\omega) = \frac{K}{(j\omega)^3(j\omega+10)} = \frac{K}{-j\omega^3(j\omega+10)} = \frac{K}{\omega^4 - j\omega^3 10}$$

$$|G(j\omega)| = \frac{K}{\omega^3\sqrt{\omega^2+100}} \quad \& \quad \angle G = -\arctan\left(\frac{-10}{\omega}\right) = \tan^{-1}\left(\frac{10}{\omega}\right)$$

when $\omega \to 0$, $|G(j0)| \to \infty$, $\angle G = 90^\circ$

Test: when $\omega = 1$, $|G(j)| \approx \frac{K}{10}$, $\angle G = 5.71^\circ$

when $\omega \to \infty$, $|G(j\infty)| \to 0$, $\angle G = 0^\circ$

(symmetry)

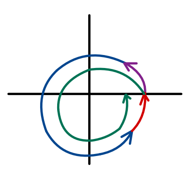
$s = Re^{j\theta}$ & $R \to \infty$, $90 \xrightarrow{\theta} -90^\circ$

$$G(Re^{j\theta}) = \frac{K}{R^3 e^{j3\theta}(Re^{j\theta}+10)} \approx K\frac{e^{-4j\theta}}{R^4} \equiv 0 \cdot e^{-4j\theta}$$

since $R \to \infty >>> 1$

For $e^{-j4\theta}$, will spin twice around origin since $90 \Rightarrow -90^\circ$ is 1 semicircle, so 4 times is twice!

$\angle -4\theta$	@ 90°	is	-360°	or	0°
	80°	is	-320°	or	40°
	⋮				
	10°	is	-40°	or	320°
	0°	is	0	or	0°
	⋮				
	-90°	is	0	or	0°

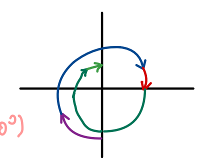


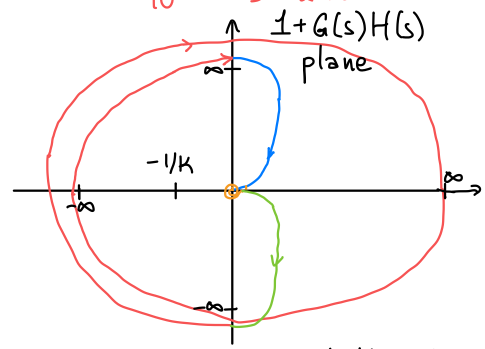
$s = re^{j\theta}$ & $R \to 0$ $-90 \xrightarrow{\theta} 90$

$$G(re^{j\theta}) = \frac{K}{r^3 e^{3j\theta}(re^{j\theta}+10)} \equiv K\frac{e^{-3j\theta}}{10\, r^3} \equiv \infty e^{-3j\theta}$$

since $10 >>> r \to 0$

$\angle -3j\theta$:	-90°	is	270°
	-80°	is	240°
	⋮		
	-10°	is	30°
	0°	is	0°
	⋮		
	80°	is	-240° (120°)
	90°	is	-270°





2 CW rotations around $-1/K$, $N=-2$

$Z = P - N = 0 - (-2) = 2 > 0$

There's 2 poles in RHP of closed loop system.

Always unstable. No values of K grant stability.

4) Controller

For a unity feedback system, the open loop transfer function is $G(s) = K[s(s+10)]^{-1}$.

a) What is the value of K where the system is critically damped? 25
b) For the K you found in part a), find the gain margin and phase margin for both an open loop response and the unity feedback.
c) Implement a controller so that
   - Ramp input steady state error is zero
   - Peak time is 500 ms when the damping factor is $1/\sqrt{2}$

and draw a simple root locus for the closed loop system that has the controller. The locus needs not be polished, but do draw a line for where the damping factor is $1/\sqrt{2}$. You don't need to find the gain of the controller; simply find its essential roots.
d) Should your controller worsen the phase margin, or improve it? Why?

a) $G(s) = \frac{K}{s(s+10)}$ closed loop

$H(s) = \frac{K}{s^2+10s+K} \equiv \frac{K}{(s+5)^2}$

$\rightarrow (s+a)^2 \rightarrow$ so $K=25$

Open loop response:

DC gain: $G(s) = 25 \frac{1}{s10(1+\frac{s}{10})}$

$\rightarrow |G(s=0)| = 2.5$

	$\lvert A \rvert$
ϕ_M: @ $1\frac{rad}{sec}$	$20\log 2.5$
$1\frac{rad}{sec}$ to 10	$20\log 2.5 - 20(\log \omega)$

$20\log 2.5 - 20(\log\omega) = 0$

so $\omega = 2.5$

so $\omega_{Gx} = 1+\omega = 3.5 \, rad/sec$

$\phi|_{\omega_{Gx}} = -90° - 45\log(2.5) = -107.91°$

$\phi_M = 180° - 107.91° = 72.09° = \phi_M$

G_M: $|A(\omega=10)| = 20(\log 2.5 - 1)$

$|A(\omega>10)| = 20(\log 2.5 - 1) - 40\log(\frac{\omega}{10})$

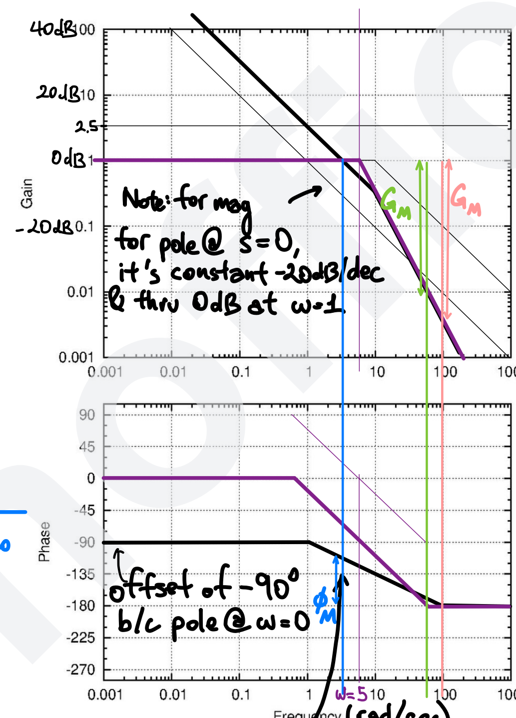
$G_m = -|A(\omega=100)| = (20(\log 2.5 - 1) - 40\log 10)(-1)$

$= +52.04\,dB$

In unity: $|H(s=0)| = \frac{K}{K} = 1$ so $\omega_{Gx} = 0$

From Bode: $G_m = 40dB$

$\phi_M = 180°$
(or $90°$, anywhere you can define)



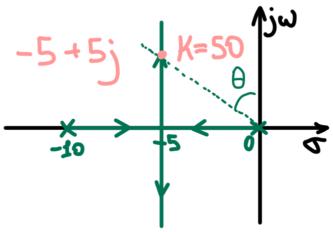
Open loop $\phi_M \simeq 72.09°$
Open loop $G_M \simeq 52.04$ dB
Closed loop: $90° \leq \phi_M \leq 180°$
Closed loop $G_M = 40dB$

Extra space for question 4)

c) $T_p = 0.5^s$ & $\zeta = 1/\sqrt{2}$

$$\omega_d = \frac{\pi}{0.5} = 2\pi \qquad \theta = \pi/4\ (45^\circ) \quad (\downarrow \arccos\zeta)$$

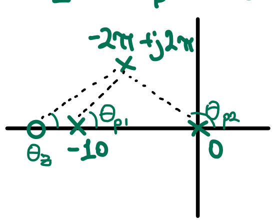
Root locus should look like



K=50 b/c $(-5+5j)\times(-5-5j)$
$= 25+25 = 50$

New pole: $\sigma = \frac{\omega_d}{\tan\theta} = \frac{2\pi}{\tan\left(\frac{\pi}{4}+\frac{\pi}{2}\right)} = -2\pi \rightarrow s = -2\pi + j2\pi$

$$\angle\theta_z - \angle\theta_p = 180^\circ = \underbrace{-\theta_{p1} - \theta_{p2}}_{\text{orig. poles}} + \underbrace{\theta_z}_{\text{PD}} = -180^\circ$$



$$\theta_z = 135^\circ + \arctan\left(\frac{2\pi}{10-2\pi}\right) - 180^\circ = 14.39^\circ$$

$$\arctan\left(\frac{\omega_p}{\sigma - \sigma_p}\right) = 14.39^\circ \rightarrow \sigma = \frac{2\pi}{\tan 14.39^\circ} + 2\pi \approx 30.77$$

$$G_{PD} = s + 30.77$$

$$G(s) = \frac{K}{s(s+10)} K_p \frac{(s+0.1)(s+30.77)}{s}$$

Is PI necessary?

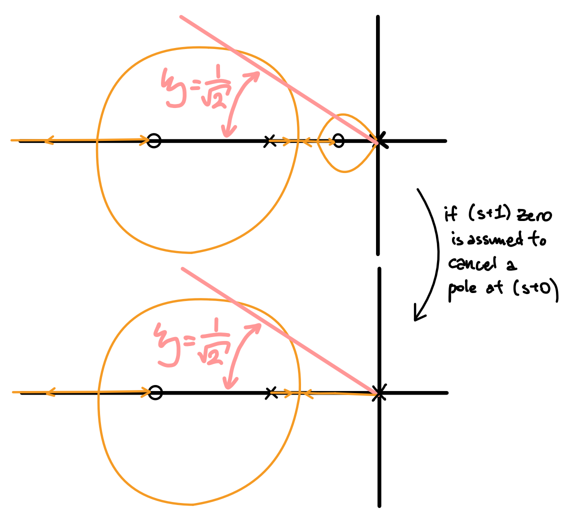
$$e_{ramp}(\infty) = \lim_{s\to 0} s \frac{R(s)}{1+G_T(s)} = \lim_{s\to 0} \frac{\cancel{s}}{s^{\cancel{2}}} \frac{1}{1+\frac{K}{s^2+10s}}$$

$$= \lim_{s\to 0} \frac{1}{\cancel{s}} \frac{s^{\cancel{2}}+10\cancel{s}}{s^2+10s+K} = \lim_{s\to 0} \frac{s+10}{s^2+10s+K} = \frac{10}{K}$$

$e_{ramp}(\infty) \neq 0$, YES, PI in necessary.

Let's choose $PI = \frac{s+0.1}{s}$

$$G_{PID}(s) = \frac{(s+0.1)(s+30.77)}{s} \times K_p$$



d) PD (and PID)'s add zeroes. so the controller improves ϕ_M.

Formula Sheet

$f(t)$	$F(s)$
$\delta(t - T)$	e^{-Ts}
$(t - T)\, u(t - T)$	$e^{-Ts} s^{-2}$
$e^{-a(t - T)}\, u(t - T)$	$e^{-Ts}\,(s + a)^{-1}$
$t^{n-1}\, e^{-at}\, u(t)$	$(n - 1)!\,(s + a)^{-n}$
$\cos(\omega t)\, e^{-at}\, u(t)$	$(s + a)\,((s + a)^2 + \omega^2)^{-1}$
$\sin(\omega t)\, e^{-at}\, u(t)$	$\omega\,((s + a)^2 + \omega^2)^{-1}$
$d_t^n f(t)$	$s^n F(s) - \sum_{i=1}^{n} s^{n-i} d_t^i f(0^-)$
$\int f(t)\, dt$	$F(s) / s$
$Ae^{-at}\, t^{n-1}\, u(t) / (n - 1)!$	$A / (a + s)^n$

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